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On Newton's Interpolation Polynomials

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In this paper functional analytic methods for nuclear locally convex spaces are applied to problems of analytic functions. The question is discussed whether the so-called Newton interpolation polynomials constitute a Schauder-basis in the space of analytic functions on the open unit circle (see Markuševič [3]). There are several different approaches to this problem, see, for instance, Walsh [7] and Gelfond [1]. Here we give a necessary and sufficient condition in terms of the interpolation points only. We consider the above space of analytic functions as a nuclear Kö-the-sequence space and use some deep theorems about nuclear spaces, such as the theorem of Dynin and Mitjagin (see Rolewicz [6], Pietsch [5]). An interesting connection with the theory of uniformly distributed sequences is mentioned.

1

We start with some general considerations:

Let $(a_{m,n})_{m,n\in\mathbb{N}_0}$ be a real, strictly positive valued infinite matrix such that $a_{m+1,n} \geqslant a_{m,n}$ for $m, n = 0, 1, 2,\ldots$. The Köthe-space $l^p[a_{m,n}]$ $(1 \leqslant p \leqslant \infty)$ is the space of all complex sequences $\xi = (\xi_n)_{n=0}^\infty$ for which $\|\xi\|_m = (\sum_{n=0}^\infty a_{n,m} |\xi_n|^p) < \infty$ for $1 \leqslant p < \infty$ and $\|\xi\|_m = \sup_n a_{m,n} |\xi_n| < \infty$ for $p = \infty$, with a topology given by the norms $\|\cdot\|_m$.

The Köthe-space $l^p[a_{m,n}]$ is nuclear if and only if for each $m \in \mathbb{N}_0$ there exists an integer $k \in \mathbb{N}_0$ such that

$$\sum_{n=0}^{\infty} \frac{a_{m,n}}{a_{m+k,n}} < \infty.$$

(See Mitjagin [4].) If this condition is fulfilled, then the space $l^1[a_{m,n}]$ is identical with the space $l^\infty[a_{m,n}]$ (see Rolewicz [6]). We restrict our interest to the space $l^1[a_{m,n}]$. A biorthogonal system $\{e_i, f_i\}$, where $e_i = (e_{i,n})_{n=0}^\infty \in l^1[a_{m,n}]$, $f_i = (f_{i,n})_{n=0}^\infty \in (l^1[a_{m,n}])'$, and $f_i(e_j) = \delta_{i,j}$ is complete, if the finite linear combinations of the sequences $\{e_i\}$ are dense in $l^1[a_{m,n}]$.

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PROPOSITION. *Let $l^1[a_{m,n}]$ be a nuclear Köthe-space. A complete biorthogonal system $\{e_i, f_i\}$ constitutes a Schauder-basis for $l^1[a_{m,n}]$ if and only if, for each $m \in \mathbb{N}_0$, there exists an integer $l \in \mathbb{N}_0$ such that*

$$\sup_{i,k} \frac{|f_{i,k}|}{a_{l,k}} \| e_i \|_m < \infty. \tag{1}$$

The proof of this proposition follows immediately from Theorem 1 in [2] and depends on the theorem of Dynin and Mitjagin, which says that each Schauder-basis of a nuclear (F)-space is absolute (i.e., the basis expansion converges absolutely).

2

Let E_1 be the space of analytic functions on the disk $|z| < 1$. We define the topology of E_1 by the norms $\|f\|_m = \sup\{|f(z)|: |z| = \exp(-1/m)\}$ for $f \in E_1$ and $m \in \mathbb{N}_0$. Let $a_{m,n} = \| z^n \|_m = \exp(-n/m)$. Then E_1 is isomorphic to $l^\infty[a_{m,n}]$, the isomorphism given by the formula $T(\sum_{n=0}^{\infty} c_n z^n) = (c_n)_{n=0}^{\infty}$.

It is easily seen that $l^\infty[a_{m,n}]$ is a nuclear space; thus we may say that E_1 is isomorphic to $l^1[a_{m,n}]$.

Now we consider Newton's interpolation polynomials. Let $(x_n)_{n=0}^{\infty}$ be a sequence of points lying in the disk $|z| < 1$. We define:

$$p_0(z) = 1, \qquad p_1(z) = (z - x_1), \dots, p_n(z) = (z - x_1)(z - x_2) \cdots (z - x_n),$$

and so on. The corresponding sequences in $l^1[a_{m,n}]$ are given by

$$\begin{aligned}
e_0 &= (1, 0, 0, \dots),\\
e_1 &= (-x_1, 1, 0, 0, \dots),\\
e_2 &= (x_1x_2, -(x_1 + x_2), 1, 0, 0, \dots),\\
e_3 &= (-x_1x_2x_3, x_1x_2 + x_1x_3 + x_2x_3, -(x_1 + x_2 + x_3), 1, 0, 0, \dots),
\end{aligned}$$

and so on. We write $e_i = (e_{i,n})_{n=0}^{\infty}$ and observe that the elements $e_{i,n}$ are the elementary symmetric functions of the variables $x_1, \dots, x_i$ for $i > n$, and that they are 1 for $i = n$ and zero for $i < n$.

The biorthogonal system of functionals for the vectors e_i is given by

$$\begin{aligned}
f_0 &= (1, x_1, x_1^2, x_1^3, x_1^4, \dots),\\
f_1 &= (0, 1, x_1 + x_2, x_1^2 + x_1x_2 + x_2^2, x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3, \dots),\\
f_2 &= (0, 0, 1, x_1 + x_2 + x_3, x_1^2 + x_1x_2 + x_1x_3 + x_2x_3 + x_2^2 + x_3^2, \dots),
\end{aligned}$$

and so on. Setting $f_i = (f_{i,k})_{k=0}^{\infty}$, we observe that $f_{i,k}$ is zero for $k < i$, $f_{i,k}$ is 1 for $k = i$ and $f_{i,k}$ is the sum of all possible (also mixed) $(k - i)$th powers of the elements $x_1, \ldots, x_{i+1}$ for $k > i$.

We can now state the main result.

THEOREM. *Newton's interpolation polynomials constitute a Schauder-basis in the space E_1 if and only if, for the interpolation points $(x_n)_{n=1}^{\infty}$, the following conditions holds: For each $m \in \mathbb{N}_0$ there exists an integer $l \in \mathbb{N}_0$ such that*

$$\sup_{i,k} \left(|f_{i,k}| \exp\left(\frac{k}{l}\right) \sum_{n=0}^{i} |e_{i,n}| \exp\left(-\frac{n+1}{m}\right)\right) < \infty. \tag{2}$$

By the above remarks, this theorem is an immediate consequence of the proposition of part 1.

Let $(x_n)_{n=1}^{\infty}$ be a null-sequence in the disk $|z| < 1$ and let $m \in \mathbb{N}_0$ be a fixed index. For each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ with $|x_n| < \epsilon$ for $n > N$; further, we assume $\sup_n |x_n| = \rho < 1$.

For $f_{i,k}$ this yields

$$|f_{i,k}| \leqslant \binom{k}{i} \rho^{k-i} \qquad \text{for } i \leqslant N,\ k \geqslant i, \tag{3}$$

$$|f_{i,k}| \leqslant \sum_{h=0}^{k-i} \binom{N+k-i-1-h}{k-i-h}\binom{i-N-1+h}{h} \rho^{k-i-h}\epsilon^{h} \qquad \text{for } i > N. \tag{4}$$

On the other hand, we have

$$\sum_{n=0}^{i} |e_{i,n}| \exp\left(-\frac{n+1}{m}\right) \leqslant \exp\left(-\frac{1}{m}\right)\left(\rho + \exp\left(-\frac{1}{m}\right)\right)^{N}\left(\exp\left(-\frac{1}{m}\right) + \epsilon\right)^{i-N}. \tag{5}$$

By some combinatorial computations, one shows that (2) holds, i.e., Newton's interpolation polynomials constitute a Schauder-basis in the space E_1.

If one of the interpolation points has absolute value 1, then Newton's interpolation polynomials are not a Schauder-basis. To show this, we may assume $|x_1| = 1$. A sequence $f = (f_n)_{n=0}^{\infty}$ is an element of the dual space $(l^1[a_{m,n}])'$ if and only if there exists an $l \in \mathbb{N}_0$ such that

$$\sup_k (|f_k|/a_{l,k}) < \infty. \tag{6}$$

In our case, this yields for the sequence $f_0 = (1, x_1, x_1^2, x_1^3, \ldots)$,

$$\sup_k (|f_{0,k}|/a_{l,k}) = \sup_k \exp(k/l) = \infty \qquad \text{for each } l \in \mathbb{N}_0.$$

If the interpolation points are uniformly distributed on a circumference $|z| = r < 1$, then the corresponding Newton's interpolation polynomials constitute a Schauder-basis, i.e., (2) is satisfied for uniformly distributed sequences (see Walsh [7]).

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